Week 3 (Lecture 2)

Computational Methods

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Overview

- We already know the nature of PDE's, now we will attempt to discretize and compute the PDE's.
- We will focus on model problems.
- These model problems have difficulties that are shared with more realistic models, but much simpler to handle.

Overview (cont'd)

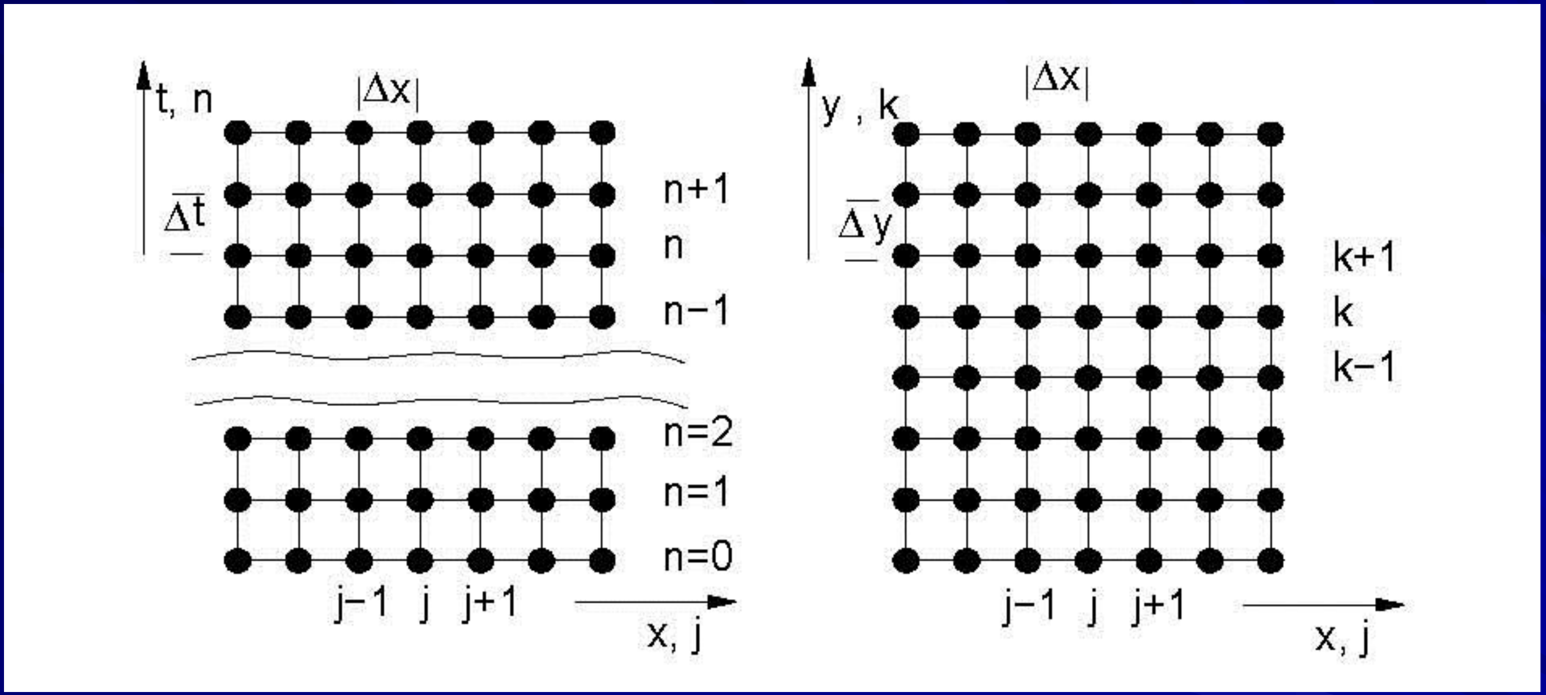
- The model problems that will be discussed are

  - $u_t + a u_x = 0$ hyperbolic
  - $u_t = \kappa u_{xx}$ parabolic
  - $u_{xx} + u_{yy} = 0$ elliptic

Overview (cont'd)

- We will discretize the models using FD method.
- Only simple uniform grids are used.
- Stick to the notations introduced before.

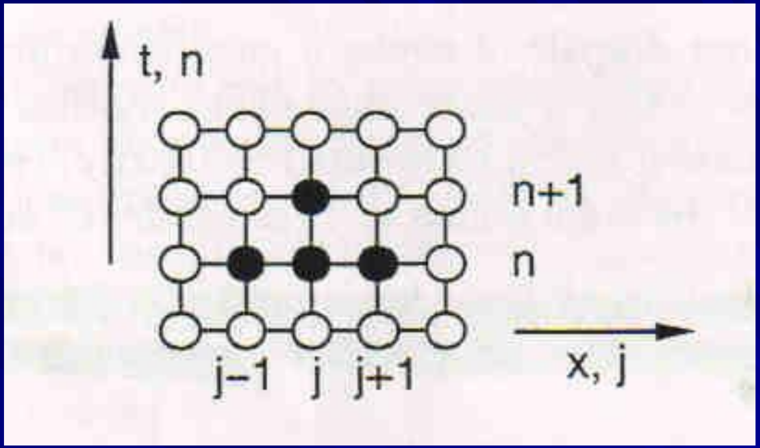
Overview (cont'd)



Evolution Equilibrium

Computation of Parabolic Equation

$$u_t = \kappa u_{xx}$$



- Apply FTCS scheme

$$u_t \approx \frac{u_j^{n+1} - u_j^n}{\Delta t}$$

$$u_{xx} \approx \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

$$u_j^{n+1} = u_j^n + \frac{\kappa \Delta t}{\Delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

von Neumann (VN) Analysis

- A method to determine stability of numerical schemes
- Decompose solution in terms of Fourier modes

$$u_j^n = \mathrm{Real}(g^n \exp(ij\theta))$$

- Variation in space in terms of sine wave from θ=[0,π]
- g is amplification factor, indication for stability
- Even though it only applies for linear problems, it provides a reasonably accurate guide to more general cases.

Von Neumann Analysis on FTCS

- Rewrite FTCS scheme solving 1D heat equation

$$u_j^{n+1} = u_j^n + \mu(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

- Using von Neumann analysis

$$g^{n+1}\exp(ij\theta) = g^n\exp(ij\theta) + \mu g^n(\exp(i(j+1)\theta) - 2\exp(i(j)\theta) + \exp(i(j-1)\theta))$$

$$\begin{aligned} g &= 1 + \mu(\exp(i\theta) - 2 + \exp(-i\theta)) \\ &= 1 + \mu(\cos\theta + i\sin\theta - 2 + \cos\theta - i\sin\theta) \\ &= 1 + 2\mu(\cos\theta - 1) \end{aligned}$$

VN on FTCS (cont'd)

- This means that if input is pure sine wave (Fourier mode), after 1 time step the sine wave will be amplified by g which depends on θ and μ.

- Restriction must be done on μ, not on θ=[0, π].

$$-1 \leq \cos\theta \leq 1$$

- For stability requires $|g| \leq 1$, hence

$$\mu \leq \frac{1}{2}$$

Comments using FTCS on 1D Heat Eqn

- The scheme is *conditionally stable*
- If we use small Δx, we need extremely small Δt since it is scaled in Δx^2
- It can be shown that the restriction

$$\mu \leq \frac{1}{2}$$

applies to all numerical schemes solving the 1D heat eqn

FTCS Solution on 1D Heat Eqn

- The electric blanket problem-heat added in the center of blanket

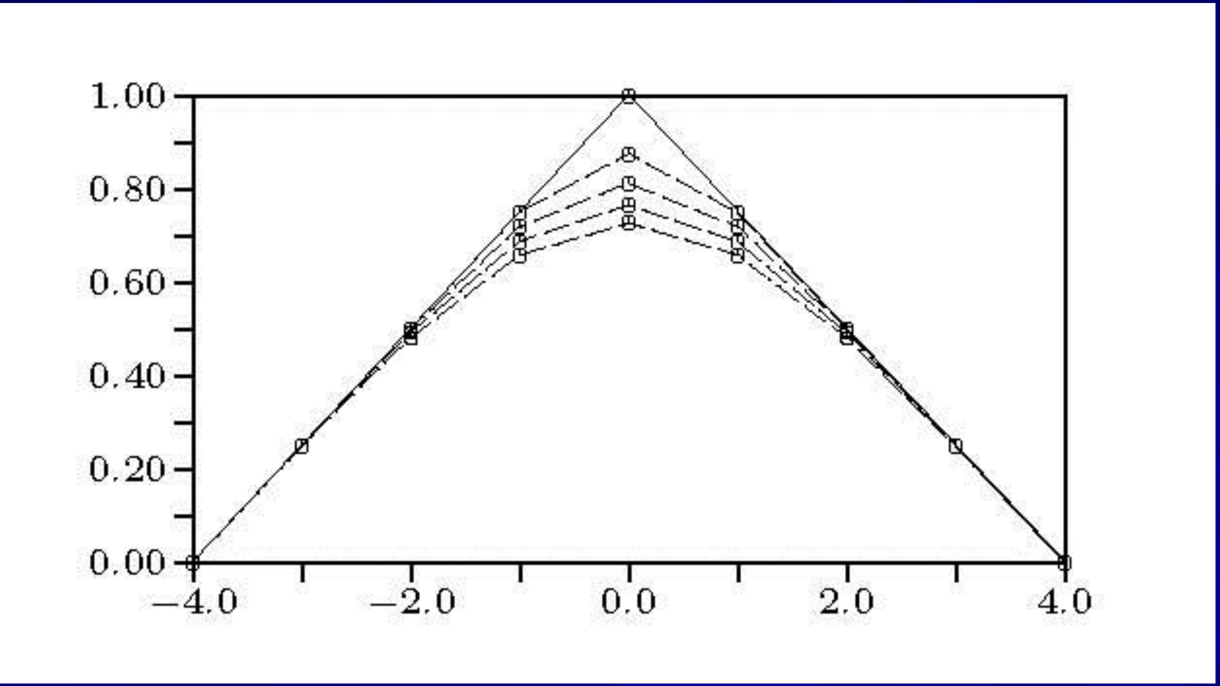
- Use 1D model where at t=0, add heat such that

$$u(x) = (1 + x), \text{if}(-1 \le x \le 0)$$

$$u(x) = (1 - x), \text{if}(0 \le x \le 1)$$

- And switch off power

FTCS Solution on 1D Heat Eqn (cont'd)

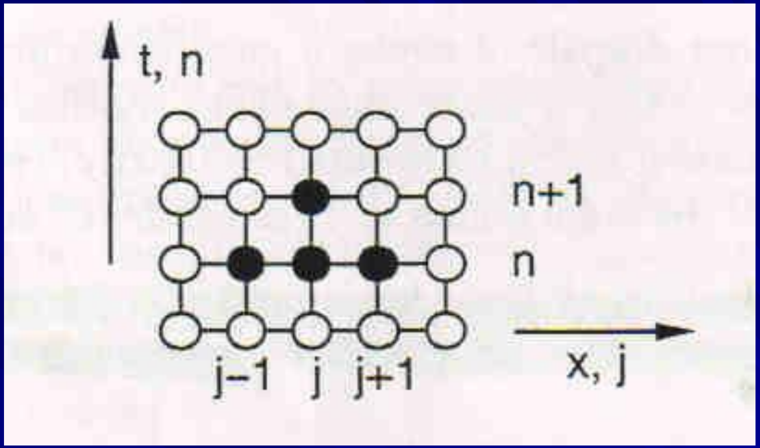


- Results are very accurate but oscillations will grow wild if restriction is violated.

$$\mu > \frac{1}{2}$$

Computation of Hyperbolic Equation

$$u_t + a u_x = 0$$



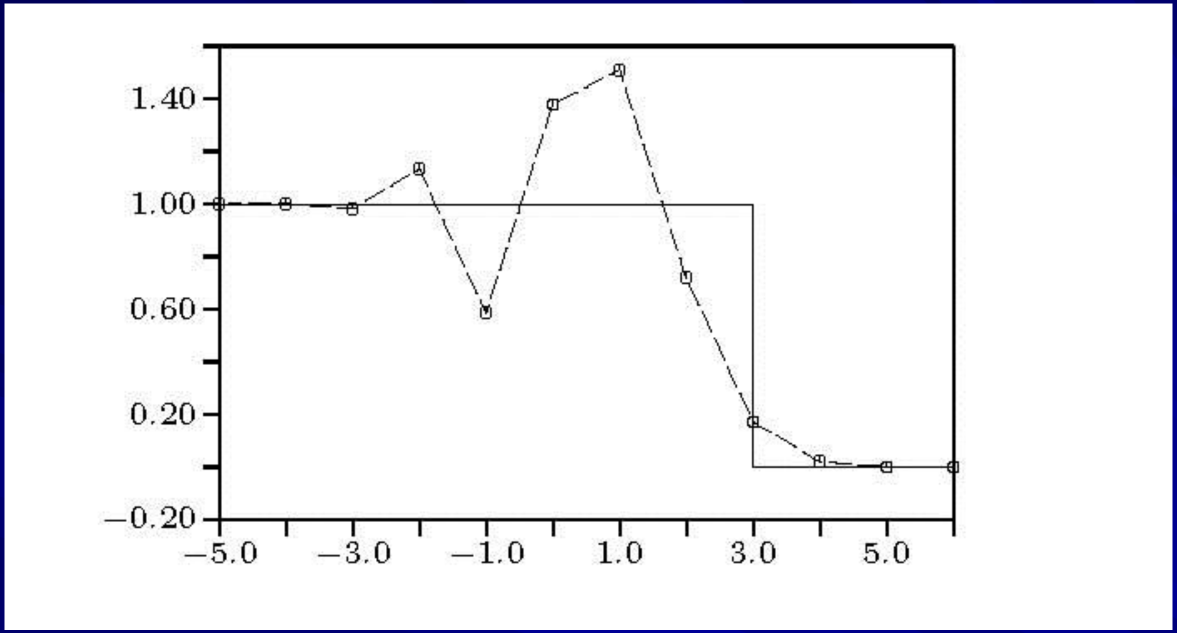
- Apply FTCS scheme

$$u_t \approx \frac{u_j^{n+1} - u_j^n}{\Delta t}$$

$$u_x \approx \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n)$$

FTCS Solution on 1D Advection Eqn



- FTCS is unstable
- Just because it works for 1 type of PDE, does not mean it will work for other types

VN Analysis on FTCS (advection)

- Rewrite FTCS scheme solving 1D heat equation

$$u_j^{n+1} = u_j^n + 0.5\nu(u_{j+1}^n - u_{j-1}^n)$$

- Using von Neumann analysis

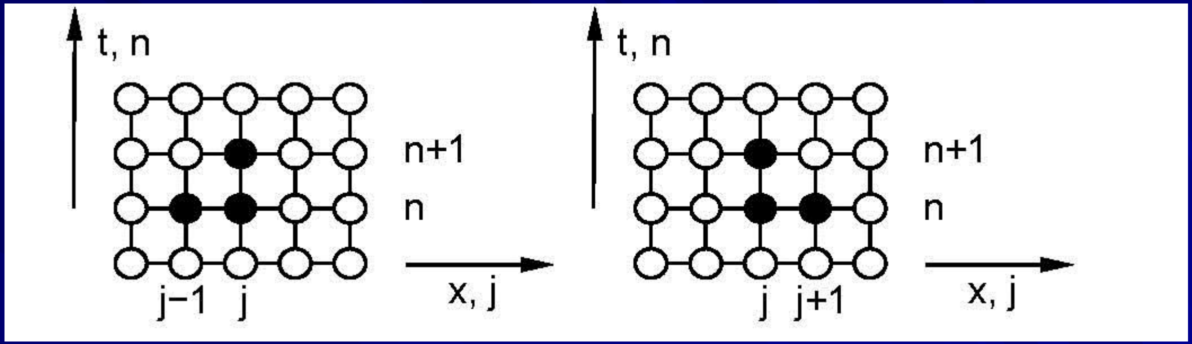
$$g^{n+1}\exp(ij\theta) = g^n\exp(ij\theta) + 0.5\mu g^n(\exp(i(j+1)\theta) - \exp(i(j-1)\theta))$$

$$g = 1 - 0.5\nu(\exp(i\theta) - \exp(-i\theta))$$
$$= 1 - 0.5i\nu\sin\theta$$
$$|g| = (1 + \nu^2\sin^2\theta)^{0.5}$$

FTCS unconditionally unstable!

Computation of Hyperbolic Equation

$$u_t + au_x = 0$$



- Apply 1st order upwind scheme

$$u_t \approx \frac{u_j^{n+1} - u_j^n}{\Delta t}$$

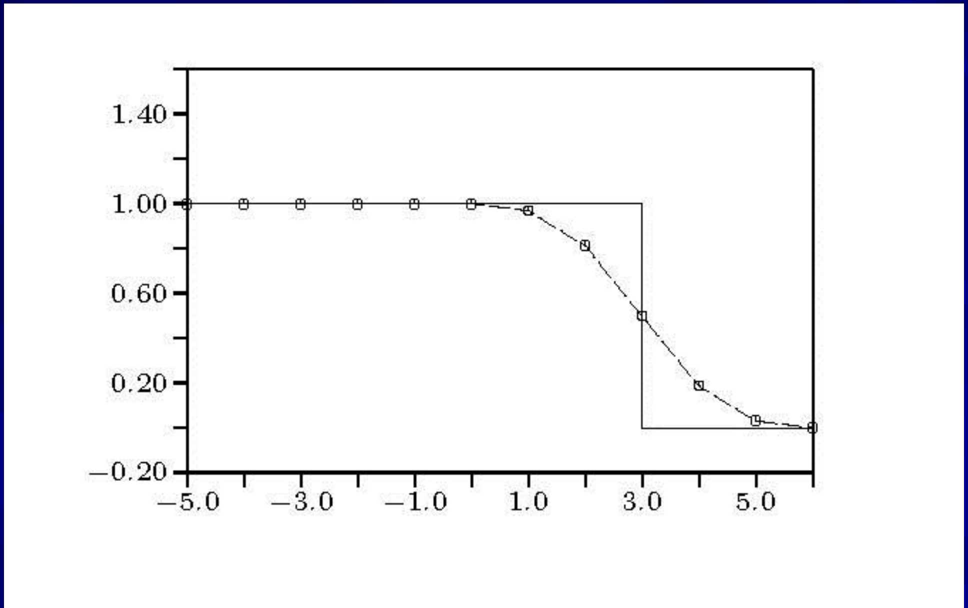
$$u_x \approx \frac{u_j^n - u_{j-1}^n}{\Delta x}, \text{if}(a > 0)$$

$$u_x \approx \frac{u_{j+1}^n - u_j^n}{\Delta x}, \text{if}(a < 0)$$

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}(u_j^n - u_{j-1}^n)$$

a > 0

1st order Upwind Solution on 1D Advection Eqn



- 1st order upwind is conditionally stable
- It can be shown using VN analysis that scheme is stable if

$$\nu \leq 1$$