Week 2 (Lecture 1)

Discretization Methods

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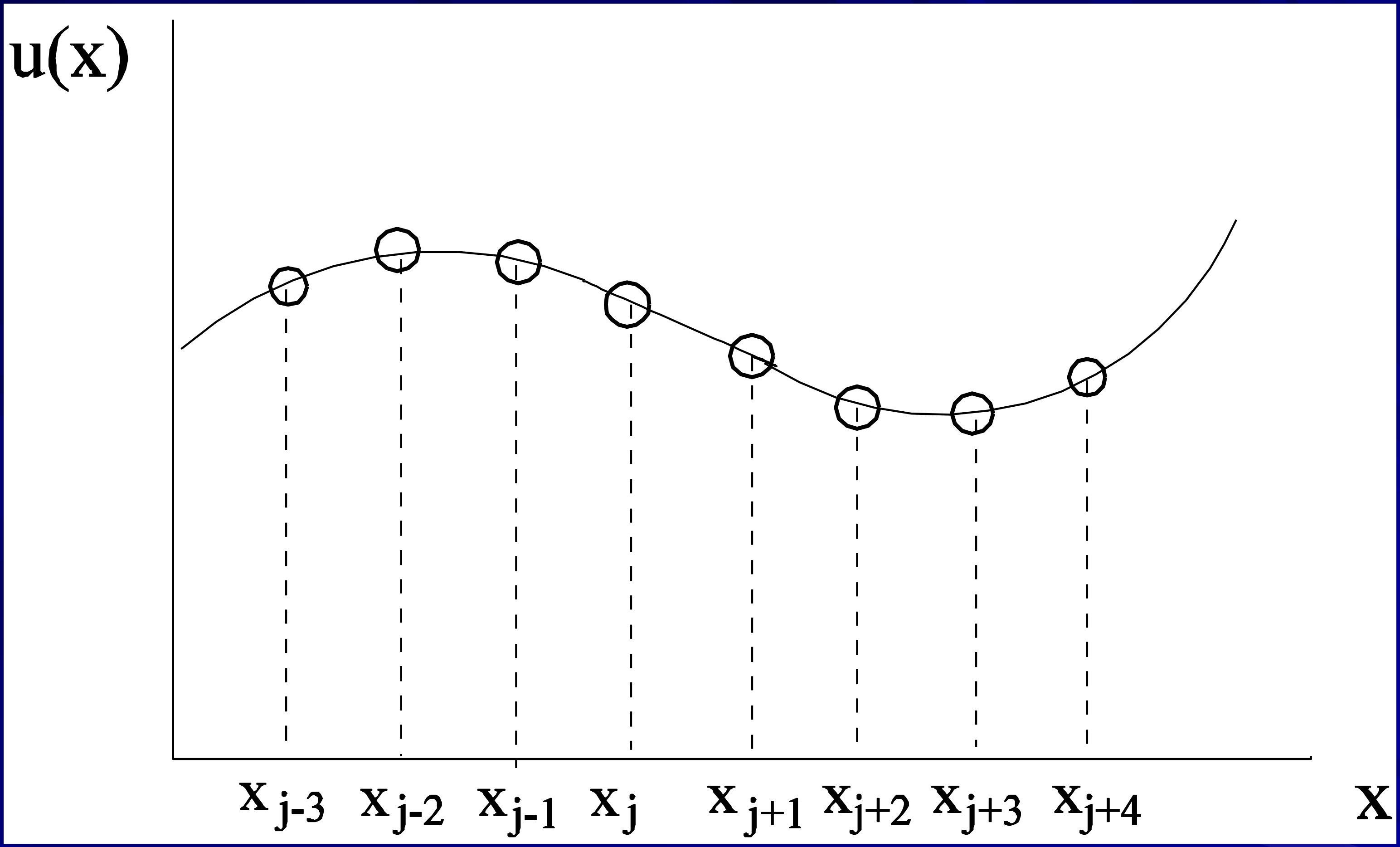
Overview

- Discuss how to represent continuous solution (data) by discrete solution.
- Can be represented by finite difference (FD), finite volume (FV), finite element (FE) or spectral methods.
- Introduce the concept of numerical errors and how to measure them.

Finite Difference (FD) Method

- A classical technique in numerical analysis.
- Assume that any set of data $\{u_j, j=1,2,..,J\}$ are samples of $u(x_j)$ from some continuous function $u(x)$
- Think of $u(x_j)$ as an attempt to reconstruct a continuous function from a fragmentary samples.

Finite Difference



$$u_j = u(x_j)$$

How to estimate error ?

Suppose we want to compute du/dx using 2 discrete points (j, j-1), the scheme would look like

$$\left(\frac{du}{dx}\right)_j \approx \frac{u_j - u_{j-1}}{h}$$

How to measure the error?

Using Taylor series analysis, it can be shown that

$$\frac{u_j - u_{j-1}}{h} = \left(\frac{du}{dx}\right)_j + h\left(\frac{d^2u}{dx^2}\right)_j + O(h^2)$$

Think of an operator on continuous and discrete data

<u>What does this tell you?</u>

Local Truncation Error (LTE)

- The difference between the numerical and analytical solution at point j is of order Δx - this is the LTE

$$L_h(P(u_h)) - L(u) = LTE$$

- LTE gives a measure of the local order of accuracy of the numerical method

- *Alternative* view of consistency: As h approaches zero, LTE approaches zero -> consistent!

LTE (cont'd)

- Can this be translated to global (convergence) sense ?
- For *linear* problems, recall that we have Lax's theorem: Consistency + Stability = Convergence (global)
- Theorem is valid regardless of type of discretization, FD, FV or Finite Element (FE), etc.
- In other words, if LTE is $O(\Delta x)$, expect $O(\Delta x)$ globally
  -> When mesh is refined, results improved at $O(\Delta x)$
  -> Usually not the case, may be better or worse

How to increase order of accuracy?

- Using more points, will *increase* accuracy of numerical method.

- Using 3 points (j, j+1, j+2)

$$\left(\frac{du}{dx}\right)_j \approx \frac{-0.5u_{j+2} + 2u_{j+1} - 1.5u_j}{h}$$

- Gives a second order accurate scheme

$$\frac{-0.5u_{j+2} + 2u_{j+1} - 1.5u_j}{h} = \left(\frac{du}{dx}\right)_j + O(h^2)$$

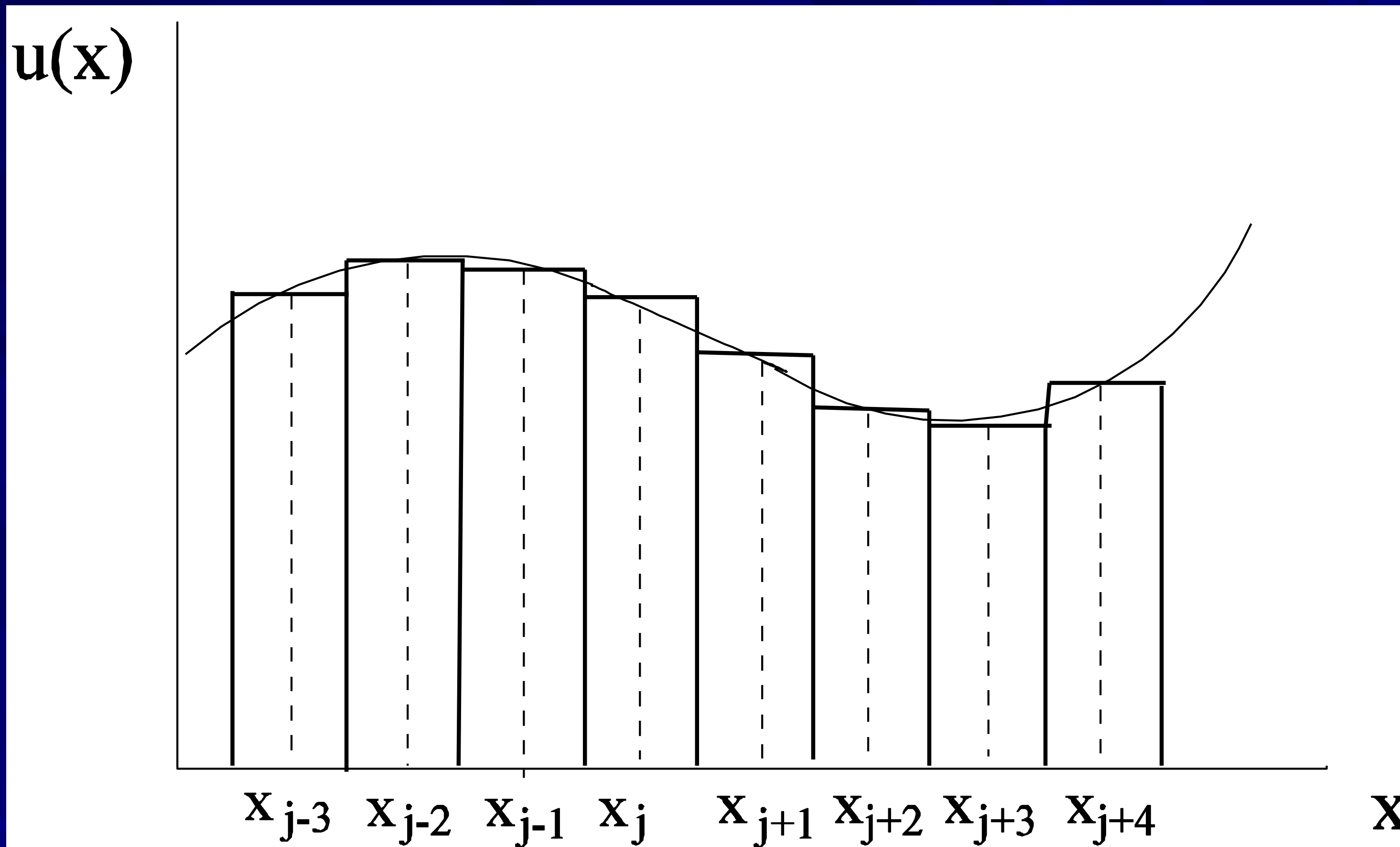
- Exercise: Find du/dx using j+1 and j-1.

Finite Volume (FV) Method

- An alternative to FD method
- Data no longer lie on a continuous curve
- Represents mean values over a set of intervals (or cells)
- Note the interfaces to left and right of jth cell will be denoted as

$$x_{j\pm\frac{1}{2}}$$

Finite Volume



$$\bar{\mathbf{u}}_j = \frac{1}{x_{j+\frac{1}{2}} - x_{j+\frac{1}{2}}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathbf{u}(x)dx$$