Week 3 (Lecture 1)

Classification of PDE

Dr. Farzad Ismail

School of Aerospace and Mechanical Engineering
Universiti Sains Malaysia
Nibong Tebal 14300 Pulau Pinang

Overview

- To create a good numerical scheme to solve PDE, we need to understand the nature of the PDE.
- We can assign PDE's into one of the 3 major categories: elliptic, parabolic and hyperbolic.
- Need to identify class of PDE either physically and/or mathematically.

Overview (cont'd)

- To distinguish whether the PDE display a wave-like behavior (hyperbolic), or
- Whether the PDE has a diffusive nature (parabolic), or
- Whether the PDE has a smooth solution development (elliptic)
- Any numerical method that ignores these questions will most likely fail!

Physical nature of Hyperbolic

- Represent a quantity that is being transported in a certain direction (i.e. a dye of ink transported in a river, traffic flow, unsteady aerodynamics, supersonic steady aerodynamics)
- Hyperbolic equations may include discontinuities such as shockwaves and contact discontinuities.
- Examples include the scalar advection equation and the wave equation.

$$u_t + a u_x = 0$$

Physical nature of Parabolic

- Represent a quantity that is being diffused or heat being conducted in omni-direction (i.e. loss of momentum in fluid due to viscosity, heat transfer due to conduction, etc)
- Parabolic equations are always smooth.
- An example include 2D diffusion problem

$$u_t = u_{xx} + u_{yy}$$

Physical nature of Elliptic

- Represent a steady state problem in which each point in the domain is affected and will affect every other point (i.e. equilibrium flight, steady heat transfer problem, steady diffusion problem, etc)

- Elliptic equations are always smooth.

- An example include 2D potential flow (Laplace equation)

$$\phi_{xx} + \phi_{yy} = 0$$

Partial Differential Equation of 2^{nd} Order

- A general scalar 2^{nd} order PDE can be represented by

$$a\frac{\partial^2\phi}{\partial x^2}+2b\frac{\partial^2\phi}{\partial x\partial y}+c\frac{\partial^2\phi}{\partial y^2}=f(\phi)$$

- If

$$b^2-4ac<0(\text{elliptic})$$
$$b^2-4ac=0(\text{parabolic})$$
$$b^2-4ac>0(\text{hyperbolic})$$

Partial Differential Equation of 2nd Order (cont'd)

- Let

$$u = \frac{\partial \phi}{\partial x}, v = \frac{\partial \phi}{\partial y}$$

- Hence the scalar 2nd order PDE now is a system of 1st order PDE

$$\begin{aligned} a\frac{\partial u}{\partial x} + 2b\frac{\partial u}{\partial y} + c\frac{\partial v}{\partial y} &= 0 \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 0 \end{aligned}$$

1st Order System of PDE

- Rewrite in matrix form

$$A\mathbf{U}_x + B\mathbf{U}_y = 0$$

- Where

$$\mathbf{U} = \begin{bmatrix} u \\ v \end{bmatrix} \qquad A = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2b & c \\ -1 & 0 \end{bmatrix}$$

- Find determinant of the following, solve for λ

$$det(B - \lambda_i A) = 0;\ \text{i=1,2}$$

1^{st} Order PDE (cont'd)

- If λ has no real roots → elliptic
- If λ has all real roots → hyperbolic
- The eigenvalues determines class of a first order system of PDE
- What about the eigenvectors?

Matrix Algebra Refresher

- Let an n system of (*well-posed*) equations represented by

$$M\mathbf{x} = b$$

- Where the right eigenvectors (**r**) satisfy the following

$$(M - \lambda I)\mathbf{r} = 0$$

- And the eigenvalues are obtained via

$$det(M - \lambda_i I) = 0$$

$$\text{i=1,2,...,n}$$

Example: 2D Potential Equation

- An inviscid, irrotational flow slightly perturbed from uniform free-stream parallel to x-axis

$$
\begin{aligned}
(1 - M_\infty^2) u_x + v_y &= 0 \\
v_x - u_y &= 0
\end{aligned}
$$

- In matrix form

$$A\mathbf{U}_x + B\mathbf{U}_y = 0$$

where

$$\mathbf{U} = \begin{bmatrix} u \\ v \end{bmatrix} \quad A = \begin{bmatrix} 1 - M_\infty^2 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

2D Potential Equation (cont'd)

- Solving for eigenvalues

$$det(A^{-1}B - \lambda_i I) = 0$$

yields

$$\lambda_1 = -\frac{1}{\sqrt{M_\infty^2 - 1}}, \lambda_2 = \frac{1}{\sqrt{M_\infty^2 - 1}}$$

What type of equation is this?

Exercise: 2D Steady Inviscid Incompressible Flow

- Show that

$$u_x + v_y = 0$$

$$uu_x + vu_y + \frac{1}{\rho}p_x = 0$$

$$uv_x + vv_y + \frac{1}{\rho}p_y = 0$$

has one real eigenvalue. What type of PDE is this?

Well-Posed Problem for PDE

- A successful combination of equations and the boundary conditions.

- The combination is well-posed if

  - A solution exists
  - The solution is unique and,
  - The solution is stable, i.e. a small change in the conditions causes only a small change in the solution.

Type of Boundary Conditions (BC)

- Inlet (Velocity or Pressure)
- Outlet
- Solid wall (hard BC)
- Periodic
- Dirichlet
- Neumann