Week 1 (Lecture 2)

Stability of ODE

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Overview

- Coding in the machines were the easy part, needed to establish mathematical principles upon which the coding is based.

- One of the most important principles is the concept of <u>stability.</u>

- Can be illustrated by solving the simple ODE problem

Find u(t) such that

$$\frac{du}{dt} = au$$

a > 0, subject to the initial condition (I.C)

$$u(0) = u^0$$

This problem is almost trivial since we know the exact analytical solution.

The analytical solution is

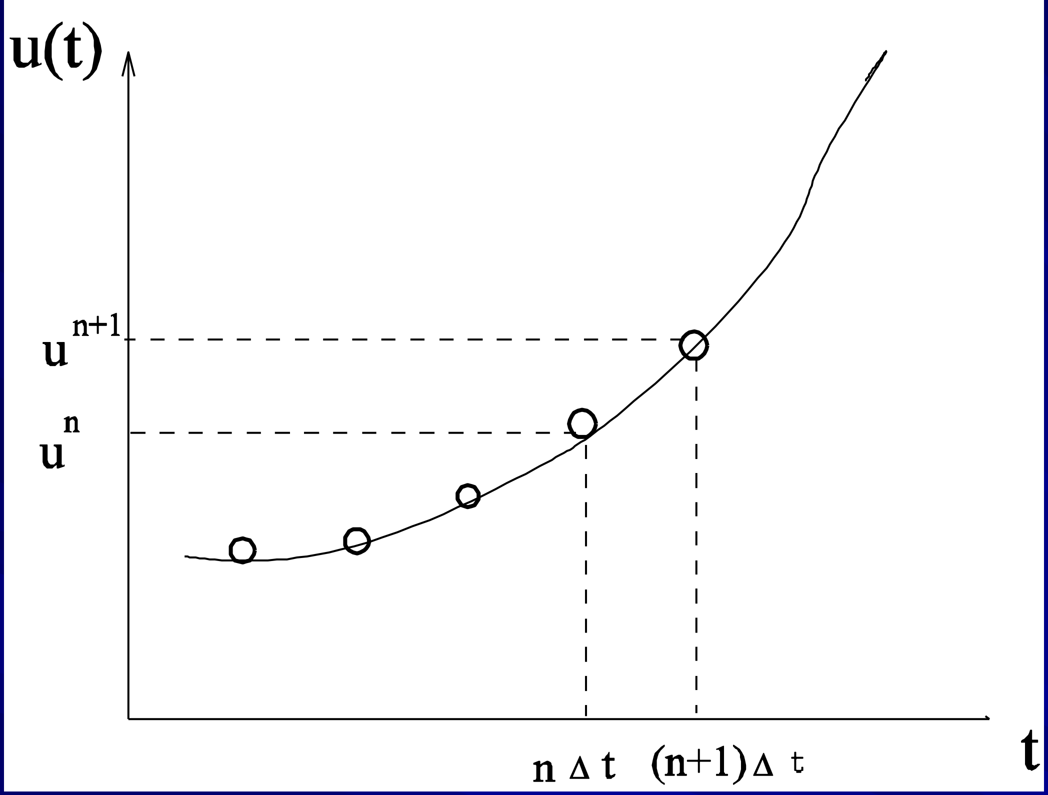
$$u(t) = u^0 \exp(at)$$

The question is what is the numerical solution?

Assume we are interested in $0<t<T$, we divide time into small intervals of Δt or

$$n = \frac{T}{\Delta t}$$

discrete time levels



$$u^n \approx u(n\Delta t)$$

It is the nth value of u, not u-to-power-n

Use Taylor series to find

$$u^{n+1} = u^n + \Delta t u_t^n + O(\Delta t^2)$$
$$\approx u^n(1 + a\Delta t)$$

This is an example of a numerical method to solve

$$\frac{du}{dt} = au$$

More generally,

$$u^{n+1} \approx u^n(1 + \alpha\Delta t^p)$$

Will this numerical method work?

It can be shown that the predicted result would be

$$u(T) = u^0(1 + \alpha\Delta t^p)^{T/\Delta t}$$

Rearrange, see what happens if we make time steps smaller,

$$u(T) = u^0(1 + \alpha T\Delta t^{p-1}\frac{\Delta t}{T})^{T/\Delta t}$$

Recall that , $\lim_{n\to\infty}(1 + a/n)^n = \exp(a)$ hence ,

$$\lim_{\Delta t\to 0} u(T) = u^0\exp[\alpha T(\Delta t^{p-1})]$$

As we decrease Δt (refinement),

- What happen if $p < 1$ and $\alpha > 0$?
- What happen if $p > 1$ and $\alpha > 0$?
- What is the correct solution?

If we choose p=1

$$\lim_{\Delta t \to 0} u(T) = u^0 \exp[\alpha T]$$

Solution is stable, but is it the correct solution?
The solution is correct (consistent) if and only if α = a

For a certain class of *linear* problems, ***Lax Equivalence Theorem says***

Stability + Consistency = Convergence

$$(p = 1) + (\alpha = a) = (\lim_{\Delta t \to 0} u(T) \to exp[aT])$$

Why would someone be so stupid to choose other than p=1 and α=a ?

- In more complicated cases, something like this might happen due to either a misconception or
- Programming error !
- The computer does not know what is wrong or right !
- It only knows to process/compute whatever is being fed in !
- <u>IT IS THE USER'S JOB</u> !

Stability versus Consistency

- A numerical scheme is unconditionally stable if its solution does not ‘blow up’ as the time-steps (or grids) are continuously being refined.

- A numerical scheme is consistent if the *local* difference between the exact solution and numerical solution approaches zero as the time steps (or grids) are refined ($h \rightarrow 0$).