EME 451-Introduction to CFD: Homework Set 2

Handed out 29/09/2015

Due 06/10/2015

1 Problem 1 - 1D Linear Advection(30%)

1.1 Revisiting the first order upwind

The 1D linear advection equation is written as

$$u_t + au_x = 0 \tag{1}$$

Using the first order upwind, the equation can be discretized as (assuming $a > 0$)

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}(u_j^n - u_{j-1}^n) \tag{2}$$

Using Taylor's series, determine the order of accuracy of this scheme for both time and space.

1.2 The q-Scheme

The linear advection equation also can be discretized using a family of numerical schemes, in this case the q-scheme.

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{q}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) \tag{3}$$

where $\nu = \frac{a\Delta t}{\Delta x}$ is a non-dimensionalized parameter called the Courant number.

1. Let $q = |\nu|$. Do you recognize this scheme?
2. Let $q = \nu^2$. Determine the order of accuracy of this scheme. Also, using von Neumann's analysis show that this scheme is stable only when $\nu \leq 1$
3. Let $q = 1$. Determine the order of accuracy of this scheme. What is the difference between this scheme and $q = |\nu|$?

2 Problem 2 - Computer Project: 1D Linear Advection (30%)

Using the family of q-schemes in problem 1 as your numerical model, write a finite-difference (FD) code to solve the 1D linear advection with the following initial conditions:

2.1 Propagation of a smooth function

Take initial conditions

$$u(x, 0) = sin(2\pi x) : 0 \leq x \leq 1$$

and periodic boundary conditions

$$u(0, t) = u(1, t)$$

Take Δx to be uniform and $a = 1$ with M=32 (repeat for M=64) grid spacings, and compute and plot solution from T=0 to a time of $T = 3/2$ using $\nu = 1.0, 0.75, 0.5, 0.25$. To implement the periodic boundary conditions, take j in your code to run from $j = 0$ to $j = 33$. At each time step, use the numerical scheme to update u_j for $1 \leq j \leq 32$ and then update

$$u_0 = u_{32}$$
$$u_{33} = u_1 \tag{4}$$

2.2 Propagation of a discontinuous function with a linear scheme

Now change initial conditions to simulate a propagating shock

$$u(j,0) = 1 : \frac{M}{4} \leq j \leq \frac{3M}{4}$$
$$u(j,0) = 0 : \text{otherwise}$$

and repeat the computation as done previously.

Discuss the results for both test cases with respect to the various choices of q.

3 Problem 3 - Computer Project: 1D Heat Equation (40%)

Unsteady heat conduction in a uniform medium in 1D is governed by

$$T_t = T_{xx} \tag{5}$$

using the dimensionless variables that make the diffusion coefficient unity.

Write a Finite-Diiference (FD) computer program to solve the "electric blanket problem" (see course notes) with uniform grid of 4,8,16, 32 using FTCS method. In each case, compute solution up to t=4 and verify that the time step is limited by

$$\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2} \tag{6}$$

In fact, you should be able to get stable results for time steps just a tiny bit bigger than this formula. Also try compute solution with negative times.

At $t = 2$, compare your coarse grid solutions (4,8,16) with your fine grid solution (32) . Do the differences compare as you would expect? (Note that in general CFD, exact solution is almost non-existence, so unless you have experimental data, most bench-markings are done using results of the finest grid).

In this case you do have the exact solution, so for extra-credit, determine the exact solution and compare the numerical with the exact solution.